the efficiency to which such shielding can be raised is a matter of importance. Considerations of mobility of the apparatus and weight of iron required necessitate the solution of the problem of maximum shielding for a given weight of iron and more than two shells, together with an examination of the limitations of utility of lamination. These problems are discussed in the paper.

2. A field of order so low as $3 \,.\, 10^{-3}$ has been obtained in a space of radius 30 cm., by the use of 1273 kgrm. of high permeability dynamo magnet steel, and an accurate method designed for the measurement of fields of lower order.

3. The leakage through air spaces in a magnetic shield has been studied, and found to be more important than is usually supposed. Although polarisation of the shells still exists to a small extent, it is not of sufficient magnitude to affect any of the conclusions which have been reached. The leakage field can, in fact, be completely isolated from other fields.

4. It is now possible to examine the behaviour of iron under practically no magnetic force.

The Period of a Spherical Resonator with a Circular Aperture.

By F. Puryer White, St. John's College, Cambridge.

(Communicated by Lord Rayleigh, O.M., F.R.S. Received June 23, 1916.)

In a paper on "The Theory of the Helmholtz Resonator,"* Lord Rayleigh has carried the determination of the wave-length of the fundamental aërial vibration in a spherical vessel with a small circular perforation to a second approximation, obtaining the result

$$\lambda = \pi\sqrt{(2S/a)} \,.\, \left\{1-\frac{9}{10}\frac{a}{2\pi c}+\ldots\right\}, \tag{1}$$

where c is the radius of the sphere, a the small radius of the aperture, and S the volume of the sphere. To obtain this value he assumes a form for the normal velocity U over the aperture and adjusts it so as to lead to agreement, to a corresponding approximation, in the values of the velocity potential ψ derived therefrom over the aperture inside and outside, so as to provide for the condition of continuity of pressure at the opening.

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* 'Roy. Soc. Proc.,' A, vol. 92, p. 265 (1916).

It is the object of this note to carry the result, by a slightly different assumption for U, to a higher approximation.

In the first place, Lord Rayleigh's equation (17) needs a slight correction; it should be replaced by

$$c^{n+1}\chi_n = C_n - iS_n,$$

where

$$C_n = \frac{1.3\ldots 2n-1}{c^n}\left\{1-\frac{c^2}{2(1-2n)}+\frac{c^4}{2.4(1-2n)(3-2n)}-\ldots\right\},$$

$$S_n = \frac{c^{n+1}}{1.3\ldots 2n+1}\left\{1-\frac{c^2}{2(2n+3)}+\frac{c^4}{2.4(2n+3)(2n+5)}-\ldots\right\}.$$

This is the notation used in the Tables recently published.*

Also from the equation (5), we have

$$\chi_0/(-c^2\chi_1) = -(1+ic)^{-1} = -1+ic+c^2-ic^3-\ldots, \tag{2}$$

which replaces (19) and (23).

We shall retain terms in c^3, which occur only in (2) above and in

$$\frac{3}{\chi_0/\chi_1-2} = \frac{3(1+ic)}{c^2-2(1+ic)} = -2+\tfrac{1}{2}-\tfrac{3}{4}c^2+\tfrac{3}{4}ic^3-\ldots, \tag{3}$$

which is used in place of (22).

The effect of these small corrections on the velocity potential is to change the term $+\frac{1}{2}c^2$ in the external value (24) to $+c^2$ and to add a term

$$ic^3\{\tfrac{3}{4}P_1(\cos\theta)-1\};$$

the internal value (29) remains unchanged.

On pp. 270–1, the determination of the constant in equation (34) may be at first sight not entirely convincing. We can verify as follows:—

$$\begin{aligned}
P_1+\tfrac{1}{2}P_2+\ldots+\frac{1}{n}P_n+\ldots &= \int_0^1 \frac{d\alpha}{\alpha}\{(1-2\alpha\cos\theta+\alpha^2)^{-\frac{1}{2}}-1\} \\
&= \int_1^\infty dy\{(1-2y\cos\theta+y^2)^{-\frac{1}{2}}-y^{-1}\} \\
&= \left[\log\left\{\frac{y-\cos\theta+(1-2y\cos\theta+y^2)^{\frac{1}{2}}}{y}\right\}\right]_1^\infty \\
&= \log 2-\log\{2\sin\tfrac{1}{2}\theta(\sin\tfrac{1}{2}\theta+1)\} \\
&= -\log\sin\tfrac{1}{2}\theta-\log(1+\sin\tfrac{1}{2}\theta).
\end{aligned}$$

This formula, although superficially different from (35), is really the same.

Substituting this result in (29) and using (24) as modified we get the following to replace (38):—

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* Doodson, 'B.A. Report,' 1914, p. 87. Note that terms have been dropped in Lord Rayleigh's expressions for $-\frac{1}{c}\frac{d}{dc}\left(\frac{\cos c}{c}\right)$, etc., on p. 268.

For a normal motion U localised over an infinitesimal area $d\sigma$ of the surface of the sphere, the difference in the values of ψ inside and outside, at a point whose angular distance from $d\sigma$ is θ, is given by

$$\psi\,(\text{in})-\psi\,(\text{out})=(\mathrm{U}d\sigma/4\pi c)\Big[2\operatorname{cosec}\tfrac{1}{2}\theta-2\log(1+\sin\tfrac{1}{2}\theta)-3(kc)^{-2}-\tfrac{9}{5}-ikc$$
$$+k^2c^2\left\{\frac{1}{175}-1+\sum_1^\infty\frac{(2n+1)\,\mathrm{P}_n(\cos\theta)}{n^2(2n+3)}+\sum_1^\infty\frac{(2n+1)\,\mathrm{P}_n(\cos\theta)}{(n+1)^2(2n-1)}\right\}$$
$$+ik^3c^3\{1-\tfrac{3}{4}\,\mathrm{P}_1(\cos\theta)\}\Big], \qquad (4)$$

as far as k^3c^3, where $k=2\pi/\lambda$.

Now the approximate result shows that θ is at most of the same order as k^2c^2; hence neglecting terms of higher order than k^3c^3,

$$\psi\,(\text{in})-\psi\,(\text{out})=(\mathrm{U}d\sigma/4\pi c)\Big[2\operatorname{cosec}\tfrac{1}{2}\theta-2\sin\tfrac{1}{2}\theta-3(kc)^{-2}-\tfrac{9}{5}-ikc$$
$$+k^2c^2\left\{\frac{1}{175}-1+\sum_1^\infty\frac{(2n+1)}{n^2(2n+3)}+\sum_1^\infty\frac{(2n+1)}{(n+1)^2(2n-1)}\right\}+\frac{1}{4}ik^3c^3\Big]. \qquad (5)$$

We have

$$\sum_1^\infty\frac{2n+1}{n^2(2n+3)}=\sum_1^\infty\left\{\frac{8}{9}\left(\frac{1}{2n}-\frac{1}{2n+3}\right)+\frac{1}{3n^2}\right\}$$
$$=\frac{8}{27}+\frac{8}{9}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots\right)+\frac{1}{3}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$
$$=\frac{8}{27}+\frac{8}{9}(1-\log_e 2)+\frac{1}{3}(\pi^2/6).$$

$$\sum_1^\infty\frac{2n+1}{(n+1)^2(2n-1)}=\sum_1^\infty\left\{\frac{8}{9}\left(\frac{1}{2n-1}-\frac{1}{2n+2}\right)+\frac{1}{3(n+1)^2}\right\}$$
$$=\frac{4}{9}+\frac{8}{9}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)+\frac{1}{3}\left(\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$
$$=\frac{4}{9}+\frac{8}{9}\log_e 2+\frac{1}{3}\left(\frac{1}{6}\pi^2-1\right).$$

Hence
$$\sum_1^\infty+\sum_1^\infty=\frac{8}{27}+1+\frac{1}{9}\pi^2=2{\cdot}39292.$$

Lord Rayleigh approximates to these series and obtains (p. 271) the value $1{\cdot}2759+1{\cdot}1178=2{\cdot}3937$.

Thus the coefficient of k^2c^2 in (5) above is

$$\frac{1}{175}-1+2{\cdot}39292=1{\cdot}39863.$$

Lord Rayleigh now proceeds by taking, at a point P of the aperture, U proportional to $\{\mathrm{CA}^2-\mathrm{CP}^2\}^{-\frac{1}{2}}$, where C is the centre of the aperture, A is a point on its circumference, and CA, CP are angular distances; this value being derived from the known electrostatic surface density on a freely-charged circular disc.

The aperture may be more strictly considered as forming part of the spherical surface, and we are thus led to consider the electrostatic density on a spherical bowl. This is proportional to

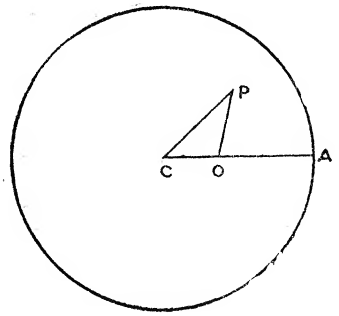
$$\left(\frac{4c^2-\mathrm{CA}^2}{\mathrm{CA}^2-\mathrm{CP}^2}\right)^{\frac{1}{3}}+\tan^{-1}\left(\frac{\mathrm{CA}^2-\mathrm{CP}^2}{4c^2-\mathrm{CA}^2}\right)^{\frac{1}{2}}\mp\tfrac{1}{2}\pi,$$

according as the point P is inside or outside, CA and CP being here the actual chords, and C being the point on the sphere diametrically opposite to the pole of the resonator.*

This suggests that, with small values of CA/c, we should assume an extra term, $\{\mathrm{CA}^2-\mathrm{CP}^2\}^{\frac{1}{2}}$, in U, and take

$$\mathrm{U}=\mathrm{A}\{\mathrm{CA}^2-\mathrm{CP}^2\}^{-\frac{1}{2}}+\mathrm{B}c^{-2}\{\mathrm{CA}^2-\mathrm{CP}^2\}^{\frac{1}{2}}.$$

The procedure is to make $\int\{\psi\,(\mathrm{in})-\psi\,(\mathrm{out})\}$ vanish as far as possible for all positions of the point O from which the angle θ is measured.



Calling the chords CA, CP, CO, and OP, a, ρ, b, and r respectively, we have

$$r=2c\sin\tfrac{1}{2}\theta,$$

$$\mathrm{U}=\mathrm{A}\,(a^2-\rho^2)^{-\frac{1}{2}}+\mathrm{B}c^{-2}\,(a^2-\rho^2)^{\frac{1}{2}},$$

and from (5) the integral which is to vanish is

$$\int d\sigma\,[4c/r-r/c-3\,(kc)^{-2}-\tfrac{9}{5}-ikc+1{\cdot}39863\,k^2c^2+\tfrac{1}{4}ik^3c^3]$$
$$[\mathrm{A}\,(a^2-\rho^2)^{-\frac{1}{2}}+\mathrm{B}c^{-2}\,(a^2-\rho^2)^{\frac{1}{2}}].\quad(7)$$

Terms of order k^4c^4 are omitted in the first bracket; hence we must neglect in the result terms of order k^6c^6 and higher. The terms independent of r depend upon

$$\int d\sigma\,\{\mathrm{A}\,(a^2-\rho^2)^{-\frac{1}{2}}+\mathrm{B}c^{-2}\,(a^2-\rho^2)^{\frac{1}{2}}\},$$

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* Maxwell, 'Electricity and Magnetism,' vol. 1, p. 279.

which is equal to

$$2\pi\int_0^a \rho\, d\rho\,\{A(a^2-\rho^2)^{-\frac{1}{2}}+Bc^{-2}(a^2-\rho^2)^{\frac{1}{2}}\} = 2\pi\{Aa+\tfrac{1}{3}Ba^3c^{-2}\}. \tag{8}$$

The remaining terms depend upon the four integrals,

$$\int\frac{d\sigma}{r}(a^2-\rho^2)^{-\frac{1}{2}},\quad \int\frac{d\sigma}{r}(a^2-\rho^2)^{\frac{1}{2}},\quad \int r\,d\sigma\,(a^2-\rho^2)^{-\frac{1}{2}},\quad \int r\,d\sigma\,(a^2-\rho^2)^{\frac{1}{2}},$$

of which the last is of order a^4 or $(kc)^8$, and must be neglected. The second and third are of order a^2, and so, if we take the first approximation, we shall only be neglecting terms of order a^3 or $(kc)^6$. Thus, considering the surface of the aperture as plane, we have

$$\rho^2 = b^2+r^2+2br\cos\phi, \tag{9}$$

ϕ being the angle POA, and the second integral is

$$\int_0^{2\pi}\int_0^{r_1} r\,dr\,d\phi\,\frac{(a^2-\rho^2)^{\frac{1}{2}}}{r} = \int_0^{2\pi}\int_0^{r_1} dr\,d\phi\,(a^2-\rho^2)^{\frac{1}{2}},$$

where r_1 is given by

$$a^2 = b^2+r_1{}^2+2br_1\cos\phi,$$

i.e.

$$r_1+b\cos\phi = (a^2-b^2\sin^2\phi)^{\frac{1}{2}}. \tag{10}$$

The integral is thus

$$\int_0^{2\pi} d\phi\left[\frac{1}{2}(r+b\cos\phi)(a^2-\rho^2)^{\frac{1}{2}}+\frac{1}{2}(a^2-b^2\sin^2\phi)\sin^{-1}\frac{r+b\cos\phi}{(a^2-b^2\sin^2\phi)^{\frac{1}{2}}}\right]_0^{r_1}$$

$$=\int_0^{2\pi} d\phi\left[\frac{1}{4}\pi(a^2-b^2\sin^2\phi)-\frac{1}{2}b\cos\phi\,(a^2-b^2)^{\frac{1}{2}}\right.$$

$$\left.-\frac{1}{2}(a^2-b^2\sin^2\phi)\sin^{-1}\frac{b\cos\phi}{(a^2-b^2\sin^2\phi)^{\frac{1}{2}}}\right]$$

$$=\frac{1}{4}\pi^2(2a^2-b^2). \tag{11}$$

The third integral is

$$\int_0^{2\pi}\int_0^{r_1} r^2\,dr\,d\phi\,(a^2-\rho^2)^{-\frac{1}{2}} = \int_0^{2\pi}\int_0^{r_1} dr\,d\phi\Big[(a^2+b^2\cos 2\phi)(a^2-\rho^2)^{-\frac{1}{2}}$$

$$-2b\cos\phi\,(r+b\cos\phi)(a^2-\rho^2)^{-\frac{1}{2}}-(a^2-\rho^2)^{\frac{1}{2}}\Big]$$

$$=\int_0^{2\pi} d\phi\,(a^2+b^2\cos 2\phi)\left(\frac{1}{2}\pi-\sin^{-1}\frac{b\cos\phi}{(a^2-b^2\sin^2\phi)^{\frac{1}{2}}}\right)$$

$$-2b\int_0^{2\pi}\cos\phi\,(a^2-b^2)^{\frac{1}{2}}\,d\phi-\tfrac{1}{4}\pi^2(2a^2-b^2)$$

$$=\pi^2a^2-\tfrac{1}{4}\pi^2(2a^2-b^2) = \tfrac{1}{4}\pi^2(2a^2+b^2). \tag{12}$$

The first integral, however, is of the order of unity, and we need a second approximation. Hence equation (9) will not do. It is evident from the formula for the electrostatic surface density on a spherical bowl, already quoted, that

$$\int \frac{d\sigma}{r}\left[\left(\frac{4c^2-a^2}{a^2-\rho^2}\right)^{\frac{1}{2}}+\tan^{-1}\left(\frac{a^2-\rho^2}{4c^2-a^2}\right)^{\frac{1}{2}}\right]$$

is proportional to the constant potential of the bowl, and in fact, from the result in the electrostatic problem as given in Maxwell, is equal to $2c\pi^2$.

Thus

$$\begin{aligned}(4c^2-a^2)^{\frac{1}{2}}\int\frac{d\sigma}{r}(a^2-\rho^2)^{-\frac{1}{2}} &= 2c\pi^2-\int\frac{d\sigma}{r}\tan^{-1}\left(\frac{a^2-\rho^2}{4c^2-a^2}\right)^{\frac{1}{2}}\\ &= 2c\pi^2-\int\frac{d\sigma}{r}\left(\frac{a^2-\rho^2}{4c^2-a^2}\right)^{\frac{1}{2}},\text{ to the order required,}\\ &= 2c\pi^2-\tfrac{1}{4}\pi^2(2a^2-b^2)(4c^2-a^2)^{-\frac{1}{2}},\text{ using (11).}\end{aligned}$$

Hence neglecting terms of higher order than a^2,

$$\int\frac{d\sigma}{r}(a^2-\rho^2)^{-\frac{1}{2}} = (4c^2-a^2)^{-\frac{1}{2}}\{2c\pi^2-\tfrac{1}{8}\pi^2c^{-1}(2a^2-b^2)\} = \pi^2+\tfrac{1}{16}\pi^2b^2/c^2. \tag{13}$$

This result has been verified by actual integration, but the work is somewhat long.

Collecting up our results, the integral (7) gives

$$\begin{aligned}&A\{4c\pi^2(1+\tfrac{1}{16}b^2/c^2)-\tfrac{1}{4}\pi^2c^{-1}(2a^2+b^2)\}+Bc^{-1}\pi^2(2a^2-b^2)\\ &\quad -2\pi a(A+\tfrac{1}{3}Ba^2c^{-2})\{3(kc)^{-2}+\tfrac{9}{5}+ikc-1{\cdot}39863k^2c^2-\tfrac{1}{4}ik^3c^3\}.\end{aligned} \tag{14}$$

This is to be zero, as far as terms of order $(kc)^5$, which gives

$$\begin{aligned}&A\{4\pi-\tfrac{1}{2}\pi a^2c^{-2}\}+B\pi c^{-2}(2a^2-b^2)\\ &\quad -2Aac^{-1}\{3(kc)^{-2}+\tfrac{9}{5}+ikc-1{\cdot}39863k^2c^2-\tfrac{1}{4}ik^3c^3\}-2Ba^3k^{-2}c^{-6}=0.\end{aligned} \tag{15}$$

Equation (15) is to hold for all values of b, hence $B=0$, and

$$3(kc)^{-2}-2\pi c/a+\tfrac{9}{5}+ikc-1{\cdot}39863k^2c^2+\tfrac{1}{4}\pi a/c-\tfrac{1}{4}ik^3c^3=0. \tag{16}$$

From this, using only the first three terms, and neglecting a^2/c^2, we get

$$kc=(3a/2\pi c)^{\frac{1}{2}}(1+9a/20\pi c), \tag{17}$$

which is Lord Rayleigh's approximation.

Substituting this in (16), we get, as far as $(a/c)^{5/2}$,

$$\begin{aligned}(kc)^{-2} &= (2\pi c/3a)\left\{1-\frac{9}{10\pi}\frac{a}{c}-i\frac{a}{2\pi c}\left(\frac{3a}{2\pi c}\right)^{\frac{1}{2}}\left(1+\frac{9a}{20\pi c}\right)\right.\\ &\qquad\left.+1{\cdot}39863\frac{a}{2\pi c}\frac{3a}{2\pi c}-\frac{1}{8}\frac{a^2}{c^2}+\frac{i}{4}\frac{a}{2\pi c}\left(\frac{3a}{2\pi c}\right)^{3/2}\right\}\\ &= (2\pi c/3a)\left\{1-\frac{9}{10\pi}\frac{a}{c}-\frac{i}{3}\left(\frac{3a}{2\pi c}\right)^{3/2}-0{\cdot}018717\frac{a^2}{c^2}-\frac{i}{60}\left(\frac{3a}{2\pi c}\right)^{5/2}\right\}.\end{aligned}$$

This gives

$$(kc)^{-1} = (2\pi c/3a)^{\frac{1}{2}} \left\{1 - \frac{9}{20\pi}\frac{a}{c} - \frac{i}{6}\left(\frac{3a}{2\pi c}\right)^{3/2} - 0{\cdot}019617\,\frac{a^2}{c^2} - \frac{7i}{120}\left(\frac{3a}{2\pi c}\right)^{5/2}\right\}, \tag{18}$$

and the wave-length is

$$\lambda = \pi(2\mathrm{S}/a)^{\frac{1}{2}}(1 - 0{\cdot}14324\,a/c - 0{\cdot}019617\,a^2/c^2 + \ldots). \tag{19}$$

Also we have

$$kc = (3a/2\pi c)^{\frac{1}{2}} \left\{1 + \frac{9}{20\pi}\frac{a}{c} + \frac{i}{6}\left(\frac{3a}{2\pi c}\right)^{3/2} + 0{\cdot}040134\,\frac{a^2}{c^2} + \frac{19i}{120}\left(\frac{3a}{2\pi c}\right)^{5/2}\right\}. \tag{20}$$

If $k = k_1 + ik_2$, the amplitude of vibration is reduced in one period in the ratio $e^{-2\pi k_2/k_1}$.

In this case

$$\begin{aligned}\frac{2\pi k_2}{k_1} &= \frac{2\pi}{6}\left(\frac{3a}{2\pi c}\right)^{3/2}\left\{1 + \frac{19}{20}\frac{3a}{2\pi c}\right\} \div \left\{1 + \frac{9}{20\pi}\frac{a}{c}\right\} \\ &= \tfrac{1}{3}\pi\left(\frac{3a}{2\pi c}\right)^{3/2}\left\{1 - \frac{39}{40}\frac{a}{\pi c}\right\}.\end{aligned} \tag{21}$$

On Some Determinations of the Sign and Magnitude of Electric Discharges in Lightning Flashes.

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The sign and magnitude of the electric charges transferred from the atmosphere to the earth in lightning flashes are of interest, not only in themselves, but also as having to be taken into account in any attempt to test whether there is a balance in the interchange of electrical charges between the atmosphere and the surface of the earth. As is well known, the evidence at present available appears to show that on the whole an excess of positive electricity passes from the atmosphere to the earth as a result of conduction by ions and convection by charged raindrops.

I am not aware of the existence of any data on which more than the roughest estimate of the magnitude of lightning discharges could be based. Schuster,* starting with two determinations by Pockels† of the maximum

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* Schuster, 'Progress of Physics (1875–1908).'

† 'Met. Zeitschrift,' p. 40 (1901).